GAUSSIAN PROCESSES
EXERCISE SHEET 11: INFINITE DIMENSIONAL MEASURES

Exercise 1 (Infinite-dimensional Lebesgue Measure)**.** *Consider*

$$\big(\mathbb{R}^{\mathbb{N}},\, \mathcal{F}_\pi,\, \mathrm{Leb}_{\otimes\mathbb{N}}\big)$$

where $\mathrm{Leb}_{\otimes\mathbb{N}}$ *is any measure satisfying, for sequences of real numbers* (a_i) *and* (b_i) *with* $a_i < b_i$*,*

$$\mathrm{Leb}_{\otimes\mathbb{N}}\Big(\prod_{i=1}^{\infty}[a_i, b_i]\Big) = \prod_{i=1}^{\infty}(b_i - a_i).$$

Recall that the product topology on $\mathbb{R}^{\mathbb{N}}$ *is metrizable; one compatible metric is*

$$d(x,y) := \sum_{n=1}^{\infty} 2^{-n}\, \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \qquad x = (x_n),\, y = (y_n) \in \mathbb{R}^{\mathbb{N}}.$$

(a) Compute

$$\mathrm{Leb}_{\otimes\mathbb{N}}\big([a,b]^{\mathbb{N}}\big).$$

(b) For $r > 0$*, let* $B(0,r)$ *denote the open ball centered at* 0 *with radius* r *with respect to the above metric* d*. Compute*

$$\mathrm{Leb}_{\otimes\mathbb{N}}\big(B(0,r)\big).$$

(c) Which natural measure-theoretic property fails for $\mathrm{Leb}_{\otimes\mathbb{N}}$*? In other words, indicate a plausible restriction (such as* σ*-finiteness) under which one concludes that a nontrivial infinite-dimensional Lebesgue measure cannot exist.*

Exercise 2 (0-1 Law for Percolation)**.** *Let* $\Gamma : \mathbb{Z}^d \to \mathbb{R}$ *be a random field such that the variables* $\{\Gamma(v)\}_{v\in\mathbb{Z}^d}$ *are i.i.d. Gaussian. For* $a \in \mathbb{R}$*, define*

$$U_a := \{v \in \mathbb{Z}^d : \Gamma(v) > a\}.$$

Consider the subgraph of $\mathbb{Z}^d$ *induced by* U_a*. Show that the event*

$$\{\textit{there exists at least one infinite connected component in the induced subgraph}\}$$

is measurable and has probability either 0 *or* 1*.*

Exercise 3 (Continuous Modification of BM)**.** *Let* $\{X_t\}_{t\in[0,1]}$ *be a stochastic process satisfying all standard properties of Brownian motion except continuity (i.e., assume* $X_0 = 0$ *and Gaussian independent increments, but do not assume continuous sample paths).*

For each $n \in \mathbb{N}$*, consider the dyadic partition of* $[0,1]$ *with mesh* 2^{-n}*. Using only the values of* X_t *at the dyadic endpoints, propose a possible explicit formula for a continuous modification* $\tilde{X}_t$ *of* X_t*.*

Discuss whether the map

$$X_\cdot \mapsto \tilde{X}.$$

that you defined is measurable. If the map is only measurable on an event of probability 1*, do you think it's enough?*